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Extensions of Simple Modules for Finite Chevalley Groups

HENNING HAAHR ANDERSEN

Matematisk Institut, Aarhus Universitet, 8000 Aarhus C, Denmark

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Let G be an almost simple algebraic group over an algebraically closed field k of positive characteristic p. We assume that G is defined and split over the prime field $\mathbb{F}_p$. For $n \geqslant 1$ we denote by $G(n)$ the finite group consisting of the points of G over the field with p^n elements.

Our first result states that for a certain class of simple modules for $G(n)$, the only extensions that can occur are those which are restrictions of G-extensions, see Theorem 2.8. The full story about G-extensions is not known yet except for a few rather small groups. However, much more is known about extensions for the algebraic group than about those for the finite group. For instance, the linkage principle [1] gives an upper bound for which G-extensions may occur. Moreover, a proof of the Lusztig conjecture on the irreducible characters for G also implies strong statements about G-extensions (see [3, Proposition 2.8]).

It is an open question whether all extensions between simple $G(n)$-modules can be found via the extension theory for G. We prove that in generic cases this is in fact so (see Theorem 3.2). Moreover, for the groups $SL(3, p)$ and $Sp(4, p)$ we are able to determine completely all extensions between simple modules from the corresponding (known) results for the algebraic groups $SL(3, k)$ and $Sp(4, k)$.

Our results extend previous results by Cline, Parshall and Scott [8, 9] (who consider extensions of the trivial module by a "minimal" module) and Smith [18] (who considers extensions between certain "minimal" modules). The method we use is a generalization of the one used by Humphreys [13] (who proves the existence of certain "selfextensions" for $Sp(4, p)$).

It should be pointed out that our methods do not work for p small. On the other hand, everything we do generalizes easily to the twisted Chevalley groups (see Theorem 3.4). We demonstrate this by working out all extensions for $SU(3, p)$ (see Section 4.3).

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1. Notation

By T we denote a split maximal torus in G and by B a Borel subgroup containing T. In the root system R associated with (G, T) we choose a set of positive roots R_+ in such a way that B corresponds to $-R_+$. We let $X(T)$ denote the character group of T and define the following subsets of this group,

$$X(T)_+ = \{\lambda \in X(T) \mid \langle \alpha^\vee, \lambda \rangle \geqslant 0, \alpha \in R_+\},$$

the set of dominant characters, and

$$X_n(T) = \{\lambda \in X(T)_+ \mid \langle \alpha^\vee, \lambda \rangle < p^n, \alpha \text{ simple root}\},$$

the set of p^n-restricted characters. Here $\alpha^\vee$ is the coroot associated to α.

Recall that $X(T)_+$ parametrizes the simple G-modules via highest weights. Here "highest" refers to the order $\leqslant$ on $X(T)$ given by $\lambda \leqslant \mu$ if $\mu - \lambda = \sum_{\alpha \in R_+} n_\alpha \alpha$ for some non-negative integers n_α. For $\lambda \in X(T)_+$ we denote by $L(\lambda)$ the simple G-module with highest weight λ. When $\lambda \in X_n(T)$, the restriction of $L(\lambda)$ to $G(n)$ remains simple, and in this way $X_n(T)$ parametrizes the simple $G(n)$-modules.

In what follows we fix $n \geqslant 1$ and decompose each $\lambda \in X(T)$ as $\lambda = \lambda^0 + p^n \lambda^1$ with $\lambda^0 \in X_n(T)$, $\lambda^1 \in X(T)$. Clearly λ^0 and λ^1 are uniquely determined by λ. If $\lambda \in X(T)_+$, then by Steinberg's tensor product theorem we have an isomorphism of G-modules

$$L(\lambda) \cong L(\lambda^0) \otimes L(\lambda^1)^{(n)},$$

where the superscript (n) denotes twist by the nth Frobenius homomorphism on G. Note that since $G(n)$ consists of the fixed points of this homomorphism, we get

$$L(\lambda) \cong L(\lambda^0) \otimes L(\lambda^1)$$

as $G(n)$-modules.

The infinitesimal subgroup scheme $G_n \leqslant G$ is defined as the kernel of the nth Frobenius homomorphism on G. The simple modules for G_n are the "same" as those for $G(n)$, namely the restrictions (to G_n) of the $L(\lambda)$'s for $\lambda \in X_n(T)$.

We let ρ denote half the sum of the positive roots. Then the Steinberg module $\mathrm{St}_n = L((p^n - 1)\rho)$ has the property that it is injective both as a $G(n)$ and as a G_n-module. When $\lambda \in X_n(T)$, we set $U_n(\lambda)$, resp. $Q_n(\lambda)$, equal to the injective $G(n)$, resp. G_n-module, whose socle is $L(\lambda)$. What we just said about the Steinberg module can then be stated as

$$\mathrm{St}_n = U_n((p^n - 1)\rho) = Q_n((p^n - 1)\rho).$$

For $p \geqslant 2(h-1)$ (h being the Coxeter number) it is known that $Q_n(\lambda)$ has a G-module structure for all $\lambda \in X_n(T)$. The easiest way to see this is by proving that the injective hull of $L(\lambda)$ in the category of p^n-bounded G-modules is injective for G_n and also has G_n-socle equal to $L(\lambda)$ (see [14]). Here we call a module p^n-bounded (following Jantzen) if all its weights μ satisfy $\langle \alpha^\vee, \mu \rangle < 2p^n(h-1)$ for all $\alpha \in R_+$. We will also say that μ is p^n-bounded if this inequality holds for μ.

It is then easy to see that $Q_n(\lambda)$ is a G-summand of (the p^n-bounded module) $\mathrm{St}_n \otimes L((p^n-1)\rho - \lambda^*)$. Here λ^* denotes the highest weight of the dual module $L(\lambda)^*$ (i.e., $\lambda^* = -w_0(\lambda)$, with w_0 being the longest element in the Weyl group). Therefore $Q_n(\lambda)$ is also injective as a $G(n)$-module. The following result, due to Jantzen [15, Corollary 2], and independently to Chastkofsky [22, Theorem 2], tells us how $Q_n(\lambda)$ splits into indecomposables for $G(n)$,

$$[Q_n(\lambda) : U_n(\mu)] = \sum_{\nu \in X(T)_+} [L(\mu) \otimes L(\nu) : L(p^n \nu + \lambda)]_G \quad \text{for all } \lambda, \mu \in X_n(T). \tag{1}$$

Here $[Q_n(\lambda) : U_n(\mu)]$ is the number of times $U_n(\mu)$ occurs as $G(n)$-summand of $Q_n(\lambda)$ and $[M : L(\omega)]_G$ is the composition factor multiplicity of the simple G-module $L(\omega)$ in M. (We use the analogous notation for $G(n)$-composition factors.)

Let α_0 denote the highest short root in R. From (1) we easily deduce

$$[Q_n(\lambda) : U_n(\lambda)] = 1 \text{ and if } \mu \neq \lambda, \text{ then } [Q_n(\lambda) : U_n(\mu)] = 0 \quad \text{unless } \langle \alpha_0^\vee, \mu \rangle \geqslant p^n - 1 + \langle \alpha_0^\vee, \lambda \rangle. \tag{2}$$

Finally we let C_0 denote the lowest alcove in $X(T)_+$; i.e.,

$$C_0 = \{\lambda \in X(T)_+ \mid \langle \alpha_0^\vee, \lambda + \rho \rangle < p\}.$$

Then the "closure" $\bar{C}_0$ of C_0 is a fundamental domain of the action of $W_p = \langle \{s_{\alpha,n} \mid \alpha \in R_+, n \in \mathbb{Z}\} \rangle$ on $X(T)$. Here $s_{\alpha,n}$ denotes the affine reflection given by $s_{\alpha,n} \cdot \lambda = s_\alpha \cdot \lambda + np\alpha = \lambda - \langle \alpha^\vee, \lambda + \rho \rangle \alpha + np\alpha$, $\lambda \in X(T)$.

2. MODULES WITH SMALL HIGHEST WEIGHTS

Assume throughout that $p \geqslant 3(h-1)$.

2.1. Let $\lambda \in X_n(T)$ and define $R_n(\lambda)$ as the G-module which makes the sequence

$$0 \to L(\lambda) \to Q_n(\lambda) \to R_n(\lambda) \to 0$$

exact.

2.2. LEMMA. *$R_n(\lambda)$ is a G-submodule of $\oplus_\nu Q_n(\nu^0) \otimes L(\nu^1)^{(n)}$, where the sum runs over all p^n-bounded weights and where the multiplicity of ν is* $\dim \operatorname{Ext}^1_G(L(\nu), L(\lambda))$.

Proof. As $Q_n(\lambda)$ is injective in the category of p^n-bounded modules, the G-socle, $\operatorname{Soc}_G R_n(\lambda)$, is given by

$$\operatorname{Soc}_G R_n(\lambda) = \bigoplus_\nu L(\nu),$$

where the sum runs over exactly those ν's stated in the lemma. We shall prove that the embedding $\operatorname{Soc}_G(R_n(\lambda)) \subsetneq \oplus_\nu Q_n(\nu^0) \otimes L(\nu^1)^{(n)}$ extends to $R_n(\lambda)$. To prove this, it is enough to verify that $\operatorname{Ext}^1_G(L(\mu), Q_n(\nu^0) \otimes L(\nu^1)^{(n)}) = 0$ for all composition factors $L(\mu)$ of $R_n(\lambda)$. As $Q_n(\nu^0)$ is injective for G_n, we have

$$\begin{aligned}
&\operatorname{Ext}^1_G(L(\mu), Q_n(\nu^0) \otimes L(\nu^1)^{(n)}) \\
&\quad \cong \operatorname{Ext}^1_{G/G_n}(L(\mu^1)^{(n)}, \operatorname{Hom}_{G_n}(L(\mu^0), Q_n(\nu^0)) \otimes L(\nu^1)^{(n)}) \\
&\quad \cong \begin{cases} \operatorname{Ext}^1_G(L(\mu^1), L(\nu^1)) & \text{if } \mu^0 = \nu^0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

But μ and ν are both p^n-bounded and our assumption on p implies that $\mu^1, \nu^1 \in \bar{C}_0$. Now the linkage principle [1] ensures that $\operatorname{Ext}^1_G(L(\mu^1), L(\nu^1)) = 0$.

2.3. LEMMA. *Let $\lambda, \mu \in X_n(T)$, and suppose ν is a weight in $\operatorname{Ext}^1_{G_n}(L(\lambda), L(\mu))$. Then there exists a simple root α such that $\nu \leqslant \lambda^* + \mu + p^{n-1}\alpha$.*

Proof. For any B-module E we denote by $H^0(E)$ the G-module induced by E. Then $L(\omega)$ is realized as the unique simple submodule of $H^0(\omega)$, $\omega \in X(T)_+$. Now set $M = H^0(\lambda^*) \otimes H^0(\mu)/L(\lambda^*) \otimes L(\mu)$. Since $\operatorname{Ext}^1_{G_n}(L(\lambda), L(\mu)) = H^1(G_n, L(\lambda^*) \otimes L(\mu))$, we have the exact sequence

$$M^{G_1} \to \operatorname{Ext}^1_{G_n}(L(\lambda), L(\mu)) \to H^1(G_n, H^0(\lambda^*) \otimes H^0(\mu)).$$

The weights of M are smaller than $\lambda^* + \mu$, so to prove the lemma, it is enough to consider the weights of $H^1(G_n, H^0(\lambda^*) \otimes H^0(\mu))$. However, $H^0(\lambda^*) \otimes H^0(\mu)$ has a filtration whose quotients have the form $H^0(\omega)$ for $\omega \leqslant \lambda^* + \mu$ (see [11] or [19]). Moreover, by [6]

$$H^1(G_n, H^0(\omega)) = \begin{cases} H^0(\omega_1)^{(n)} & \text{if } \omega = p^n\omega_1 - p^i\alpha \text{ for some} \\ & \text{simple root } \alpha \text{ and } i < n \\ 0 & \text{otherwise.} \end{cases}$$

The lemma follows.

2.4. Recall that if G is of type A_1, then all $\mathrm{Ext}^1_{G(n)}(L(\lambda), L(\mu))$ were computed in [5]. We shall therefore assume in what follows that G is not of type A_1. This ensures that $\langle \alpha_0^\vee, \alpha \rangle \leqslant 1$ for all simple roots α.

2.5. Lemma. *Let $\lambda \in X_n(T)$ and suppose ν is a p^n-bounded weight for which* $\mathrm{Ext}^1_G(L(\nu), L(\lambda)) \neq 0$. *Then* $\langle \alpha_0^\vee, \nu^0 + \lambda \rangle \geqslant p^n \langle \alpha_0^\vee, \nu^1 \rangle - p^{n-1}$.

Proof. Consider first the case where $\nu^0 = \lambda$. Then by [2, Theorem 4.5] we have $\mathrm{Ext}^1_{G_n}(L(\nu^0), L(\lambda)) = 0$, and hence

$$\mathrm{Ext}^1_G(L(\nu), L(\lambda) \cong \mathrm{Ext}^1_{G/G_n}(L(\nu^1)^{(n)}, k) = 0$$

because $\nu^1 \in \bar{C}_0$. Therefore we must have $\nu^0 \neq \lambda$. In this case,

$$\mathrm{Ext}^1_G(L(\nu), L(\lambda)) \cong \mathrm{Hom}_{G/G_n}(L(\nu^1)^{(n)}, \mathrm{Ext}^1_{G_n}(L(\nu^0), L(\lambda))).$$

By Lemma 2.3, this is 0 unless $p^n\nu^1 \leqslant \nu^{0*} + \lambda + p^{n-1}\alpha$ for some simple root α. In particular, we must have $\langle \alpha_0^\vee, \nu^0 + \lambda \rangle = \langle \alpha_0^\vee, \nu^{0*} + \lambda \rangle \geqslant p^n \langle \alpha_0^\vee, \nu^1 \rangle - p^{n-1}$.

2.6. The following proposition is part of [10, Theorem 7.4]. There the proof is based upon [20, Theorem 2D]. For completeness we include a selfcontained proof (see also [13]). No restriction on p is needed here.

2.7. Proposition. *Let $\lambda, \mu \in X_n(T)$. Then the restriction map*

$$\mathrm{Ext}^1_G(L(\mu), L(\lambda)) \to \mathrm{Ext}^1_{G(n)}(L(\mu), L(\lambda))$$

is injective.

Proof. As $\mathrm{Ext}^1_G(L(\mu), L(\lambda)) \cong \mathrm{Ext}^1_G(L(\lambda^*), L(\mu^*))$ and the same holds for $G(n)$, we may assume that $\mu < \lambda$.

Let E be a non-trivial G-extension of $L(\mu)$ by $L(\lambda)$. We shall show that $L(\mu)$ is not a $G(n)$-submodule of E.

It is easy to see that E is a G-submodule of $H^0(\lambda)$. In fact, λ is a highest weight in E, and hence there is a B-homomorphism $E \to \lambda$. This gives a G-homomorphism $E \to H^0(\lambda)$ which clearly must be injective.

As λ is a B-submodule of $(p^n-1)\rho \otimes H^0((p^n-1)\rho - \lambda^*)$ (being the minimal weight), we get $H^0(\lambda) \subset H^0((p^n-1)\rho \otimes H^0((p^n-1)\rho-\lambda^*)) \cong \mathrm{St}_n \otimes H^0((p^n-1)\rho-\lambda^*)$, so that $E \subset \mathrm{St}_n \otimes H^0((p^n-1)\rho-\lambda^*))$. It therefore suffices to show that $\mathrm{Hom}_{G(n)}(L(\mu), \mathrm{St}_n \otimes H^0((p^n-1)\rho-\lambda^*)) = \mathrm{Hom}_{G(n)}(\mathrm{St}_n, L(\mu^*) \otimes H^0((p^n-1)\rho-\lambda^*))$ is zero. To see this, let $L(\nu)$ be a G-composition factor of $L(\mu^*) \otimes H^0((p^n-1)\rho-\lambda^*)$. Then $\nu \leqslant \mu^* + (p^n-1)\rho - \lambda^* < (p^n-1)\rho$, and as a $G(n)$-module, $L(\nu) \cong L(\nu^0) \otimes L(\nu^1)$. Any $G(n)$-composition factor $L(\omega)$ of $L(\nu)$ must therefore satisfy $\langle \alpha_0^\vee, \omega \rangle \leqslant \langle \alpha_0^\vee, \nu^0 + \nu^1 \rangle \leqslant \langle \alpha_0^\vee, \nu \rangle \leqslant (p^n-1)(h-1)$. Note that equality cannot hold everywhere. In particular we conclude that $\mathrm{Hom}_{G(n)}(\mathrm{St}_n, L(\nu)) = 0$ for all such ν.

2.8. THEOREM. *Let $\lambda, \mu \in X_n(T)$ with $\langle \alpha_0^\vee, \lambda + \mu \rangle < p^n - p^{n-1} - 1$. Then the restriction map*

$$\mathrm{Ext}^1_G(L(\mu), L(\lambda)) \to \mathrm{Ext}^1_{G(n)}(L(\mu), L(\lambda))$$

is an isomorphism.

Proof. As in Proposition 2.7 we may assume that $\langle \alpha_0^\vee, \mu \rangle \leqslant \langle \alpha_0^\vee, \lambda \rangle$. From the short exact sequence

$$O \to L(\lambda) \to Q_n(\lambda) \to R_n(\lambda) \to 0,$$

together with (2) in Section 1, we obtain

$$\mathrm{Ext}^1_{G(n)}(L(\mu), L(\lambda)) \cong \mathrm{Hom}_{G(n)}(L(\mu), R_n(\lambda)).$$

By Lemma 2.2 we have $R_n(\lambda) \subset M_1 \oplus M_2$, where M_1 consists of those summands in $\oplus_\nu Q_n(\nu^0) \otimes L(\nu^1)^{(n)}$, where $\nu^1 = 0$, and M_2 consists of the rest. Note that since $\langle \alpha_0^\vee, \mu \rangle < p^n - 1$, we have via (2) from Section 1 that $\mathrm{Hom}_{G(n)}(L(\mu), M_1) \cong \mathrm{Hom}_G(L(\mu), M_1) \cong \mathrm{Ext}^1_G(L(\mu), L(\lambda))$. Hence we are done if we prove that $\mathrm{Hom}_{G(n)}(L(\mu), M_2) = 0$; i.e., we want to have

$$\mathrm{Hom}_{G(n)}(L(\mu), Q(\nu^0) \otimes L(\nu^1)) = 0$$

for all p^n-bounded weights ν for which $\nu^1 \neq 0$ and $\mathrm{Ext}^1_G(L(\nu), L(\lambda)) \neq 0$. This will certainly be the case if we check that

$$\mathrm{Hom}_{G(n)}(L(\omega), Q_n(\nu^0)) = 0$$

for all composition factors $L(\omega)$ of $L(\mu) \otimes L(\nu^{1*})$. Using Section 1, Eq. (2) once again, we see that it is enough to check that $\langle \alpha_0^\vee, \omega \rangle < \langle \alpha_0^\vee, \nu^0 \rangle$ for all such ω. But $\langle \alpha_0^\vee, \omega \rangle \leqslant \langle \alpha_0^\vee, \mu \rangle + \langle \alpha_0^\vee, \nu^1 \rangle$ and by Lemma 2.5 together with our assumption on $\langle \alpha_0^\vee, \lambda + \mu \rangle$ we get

$$\langle \alpha_0^\vee, \nu^0 \rangle \geqslant p^n \langle \alpha_0^\vee, \nu^1 \rangle - p^{n-1} - \langle \alpha_0^\vee, \lambda \rangle = (p^n - 1)\langle \alpha_0^\vee, \nu^1 \rangle - p^{n-1}$$
$$- \langle \alpha_0^\vee, \lambda + \mu \rangle + \langle \alpha_0^\vee, \nu^1 + \mu \rangle > \langle \alpha_0^\vee, \nu^1 + \mu \rangle \geqslant \langle \alpha_0^\vee, \omega \rangle.$$

Remark. The proof shows that we may replace the assumption $\langle \alpha_0^\vee, \mu + \lambda \rangle < p^n - p^{n-1} - 1$ in the theorem by $\langle \alpha_0^\vee, \mu + \lambda \rangle < c(p^n - 1) - p^{n-1}$, where $c = \min_\nu \langle \alpha_0^\vee, \nu^1 \rangle$, the minimum taken over all p^n-bounded weights ν with $\nu^1 \neq 0$ and $\mathrm{Ext}_G^1(L(\nu), L(\lambda)) \neq 0$.

2.9. Combining Theorem 2.8 with the linkage principle [1], we obtain

2.10. COROLLARY. *Let* λ, μ *be as in Theorem* 2.8. *Then* $\mathrm{Ext}^1_{G(n)}(L(\mu), L(\lambda)) = 0$ *unless* $\lambda = w \cdot \mu$ *for some* $w \in W_p \setminus \{1\}$. *In particular,* $\mathrm{Ext}^1_{G(n)}(L(\mu), L(\lambda)) = 0$ *for all* $\mu, \lambda \in \bar{C}_0$ *with* $\langle \alpha_0^\vee, \mu + \lambda \rangle < p^n - p^{n-1} - 1$.

Remark. (i) The very last condition is always satisfied for $\mu, \lambda \in \bar{C}_0$ if $n \geqslant 2$.

(ii) This corollary gives immediately that any "minimal module" is completely reducible for $G(n)$. This should be compared with [8, 18], which contain this result for certain classes of "minimal" modules (but for all primes).

(iii) If μ and λ are fixed, it follows immediately from this theorem that $\mathrm{Ext}^1_{G(n)}(L(\mu), L(\lambda)) = \mathrm{Ext}^1_G(L(\mu), L(\lambda))$ for n large.

3. THE GENERIC CASE

3.1. Let H be any group and let M, N, and P be three H-modules. Then composition of maps induces a homomorphism

$$\mathrm{Hom}_H(M, N) \otimes \mathrm{Hom}_H(N, P) \to \mathrm{Hom}_H(M, P),$$

and more generally this gives us natural homomorphisms

$$\mathrm{Ext}^i_H(M, N) \otimes \mathrm{Ext}^j_H(N, P) \to \mathrm{Ext}^{i+j}_H(M, P)$$

for every $i, j \geqslant 0$.

If we take $H = G$ and combine the above with the restriction maps $\mathrm{Ext}^i_G(M, N) \to \mathrm{Ext}^i_{G(n)}(M, N)$, $i \geqslant 0$, then we obtain, particularly for every $\lambda, \mu \in X_n(T)$, a homomorphism

$$\oplus\, \mathrm{Hom}_G(L(\mu), L(\nu^0) \otimes L(\nu^1)) \otimes \mathrm{Ext}^1_G(L(\nu), L(\lambda)) \to \mathrm{Ext}^1_{G(n)}(L(\mu), L(\lambda)), \tag{1}$$

where the sum is taken over all p^n-bounded weights ν. If we examine the proof of Theorem 2.8, we see that what we proved there was in fact that under the given restrictions on λ and μ (as well as on p) the homomorphism (1) is an isomorphism. In Theorem 3.2 we shall see that the same is true for λ and μ "sufficiently deep" inside alcoves. Our method of proof is rather crude, and in the next section we shall prove that for certain small groups the statement is in fact true for all $\lambda, \mu \in X_n(T)$. It would of course be very nice to have such a result in general since this would provide a way of computing $G(n)$-extensions from a knowledge of G-extensions. However, for groups other than those treated in Section 4, this is so far only wishful thinking.

3.2. THEOREM. *Let* $\lambda, \mu \in X_n(T)$ with $\langle \alpha_0^\vee, \mu \rangle \leqslant \langle \alpha_0^\vee, \lambda \rangle$. *Suppose* λ *and* μ *have distances at least* $2(h-1)$ (*resp.* $h-1$) *to any wall* (*i.e., if* p *divides* $\langle \alpha^\vee, \lambda+\rho \rangle + c$ (*resp.* $\langle \alpha^\vee, \mu+\rho \rangle + c$) *for some* $c \in \mathbb{Z}$, *then* $|c| \geqslant 2(h-1)$ (*resp.* $h-1$)). *Then Eq.* (1) *of Section* 3.1 *is an isomorphism.*

Proof. As in the proof of Theorem 2.8 we have

$$\operatorname{Ext}_G^1(L(\nu), L(\lambda)) \cong \operatorname{Hom}_G(L(\nu), R_n(\lambda))$$

for all p^n-bounded weights ν. Similarly,

$$\operatorname{Ext}_{G(n)}^1(L(\mu), L(\lambda)) \cong \operatorname{Hom}_{G(n)}(L(\mu), R_n(\lambda))$$

so that Eq. (1) of Section 3.1 reads

$$\begin{aligned} \oplus \operatorname{Hom}_G(L(\mu), L(\nu^0) \otimes L(\nu^1)) \otimes \operatorname{Hom}_G(L(\nu), R_n(\lambda)) \\ \to \operatorname{Hom}_{G(n)}(L(\mu), R_n(\lambda)). \end{aligned} \tag{1}$$

To prove that (1) is an isomorphism, it will be enough to check that

$$\operatorname{Hom}_G(L(\mu), L(\nu^0) \otimes L(\nu^1)) = \operatorname{Hom}_{G(n)}(L(\mu), Q_n(\nu^0) \otimes L(\nu^1)) \tag{2}$$

for all ν which contribute non-trivially to the left hand side of (1). In fact, (2) implies that $L(\mu)$ is a $G(n)$-submodule of $R_n(\lambda) \subset \oplus\, Q_n(\nu^0) \otimes L(\nu^1)$ if and only if $L(\mu)$ is a G-submodule of the G-socle of $R_n(\lambda)$ restricted to $G(n)$.

Now, to establish (2), note first that by the linkage principle [1] ν is linked to λ. Hence ν^0 also has distance at least $2(h-1)$ to the walls. Moreover, since ν is p^n-bounded, we have $|\langle \alpha^\vee, \omega \rangle| \leqslant \langle \alpha_0^\vee, \nu^1 \rangle < 2(h-1)$ for all weights ω of $L(\nu^1)$ and all $\alpha \in R$; i.e., $\nu^0 + \omega$ belongs to the same alcove as ν^0 for all such ω. The linkage principle therefore also ensures that $L(\nu^0) \otimes L(\nu^1) \cong \oplus_\omega L(\nu^0 + \omega)$ (as G-modules), where the sum is over all weights (with multiplicities) of $L(\nu^1)$. It follows that $\dim \operatorname{Hom}_G(L(\mu),$

$L(\nu^0) \otimes L(\nu^1)) = [L(\nu^0) \otimes L(\nu^1): L(\mu)]_G$ ($= \dim L(\nu^1)_{\mu - \nu^0}$, the $(\mu - \nu^0)$-weight space in $L(\nu^1)$).

Next we note that $2(p^n - 1)\rho - \nu^{0*} + \nu^1$ is the highest weight of $Q_n(\nu^0) \otimes L(\nu^1)$ and that $\langle \alpha_0^\vee, 2(p^n - 1)\rho - \nu^{0*} + \nu^1 \rangle = 2(p^n - 1)(h-1) - \langle \alpha_0^\vee, \nu^0 \rangle + \langle \alpha_0^\vee, \nu^1 \rangle < 2p^n(h-1)$ so that this module is p^n-bounded. As $\operatorname{Ext}^1_G(L(\lambda), Q_n(\nu^0)) = \operatorname{Ext}^1_{G/G_n}(L(\lambda^1), \operatorname{Hom}_{G_n}(L(\lambda^0), Q_n(\nu^0)))$ is zero unless $\lambda^0 = \nu^0$ and $\lambda^1 \geqslant (p - h + 1)\alpha_0$ (and so in particular is zero if $\langle \alpha_0^\vee, \lambda \rangle < 2(p^n + 1)(h-1)$ because $2(p^n+1)(h-1) \leqslant 2p^n(p-h+1) \leqslant \langle \alpha_0^\vee, \nu^0 + p^n(p-h+1)\alpha_0 \rangle$), we conclude that $\operatorname{Ext}^1_G(M, Q_n(\nu^0) \otimes L(\nu^1)) = \operatorname{Ext}^1_G(M \otimes L(\nu^1)^*, Q_n(\nu^0))$ is zero for all p^n-bounded modules M. In other words, $Q_n(\nu^0) \otimes L(\nu^1)$ is injective among p^n-bounded modules. Hence an argument as above gives

$$Q_n(\nu^0) \otimes L(\nu^1) \cong \bigotimes_{\omega} Q_n(\nu^0 + \omega),$$

where the sum runs over the weights of $L(\nu^1)$ (with multiplicities).

Hence to prove (2), we must show for all ω above that $\dim \operatorname{Hom}_{G(n)}(L(\mu), Q_n(\nu^0 + \omega)) = 0$ unless $\mu = \nu^0 + \omega$. Using (1) in Section 1, this is so unless there exists $\eta \neq 0$ with $[L(\mu) \otimes L(\eta): L(p^n\eta + \nu^0 + \omega)]_G \neq 0$. But if such an η exists, then $\mu + \eta \geqslant p^n\eta + \nu^0 + \omega$, and therefore $(p^n - 1)\langle \alpha_0^\vee, \eta \rangle < \langle \alpha_0^\vee, \mu \rangle \leqslant (p^n - 1)(h-1)$; i.e., $\langle \alpha_0^\vee, \eta \rangle < h - 1$. Hence by the assumption on μ, $L(\mu) \otimes L(\eta)$ is a direct sum of simple modules whose highest weights all belong to the alcove containing μ. In particular $L(p^n\eta + \nu^0 + \omega)$ is not a composition factor.

Remark. An easy modification of the above proof shows that if we require μ to have distance at least $3(h-1)$ from the walls, then the theorem holds for all λ.

3.3. Let π denote an automorphism of the Dynkin diagram for G. Then π induces a group automorphism of G which we also denote by π, and we set $G_\pi(n)$ equal to the fixed points of π composed with the nth Frobenius homomorphism on G.

It is easy to check that everything we have obtained concerning extensions for $G(n)$ generalizes to $G_\pi(n)$. The main point is that there exists an analogue of (1) in Section 1 for $G_\pi(n)$ (see [15, Corollary 2.10]. Then all our arguments above also hold for $G_\pi(n)$ because the automorphism on $X(T)$ induced by π keeps α_0 fixed. Thus we have the following analogues of Theorems 2.8 and 3.2.

3.4. Theorem. (i) *Let $\lambda, \mu \in X_n(T)$ with $\langle \alpha_0^\vee, \mu + \lambda \rangle < p^n - p^{n-1} - 1$. Then the restriction map*

$$\operatorname{Ext}^1_G(L(\mu), L(\lambda)) \to \operatorname{Ext}^1_{G_\pi(n)}(L(\mu), L(\lambda))$$

is an isomorphism.

(ii) *Let* $\lambda, \mu \in X_n(T)$ *such that* λ (*resp.* μ) *has distance at least* $2(h-1)$ (*resp.* $h-1$) *to the walls. Then*

$$\bigoplus_{\nu} \operatorname{Hom}_G (L(\mu), L(\nu^0) \otimes L(\pi(\nu^1))) \otimes \operatorname{Ext}^1_G (L(\nu), L(\lambda))$$
$$\cong \operatorname{Ext}^1_{G_{\pi(n)}} (L(\mu), L(\lambda)),$$

where the sum is over all p^n*-bounded weights* ν.

4. $SL(3, p)$ AND $SU(3, p)$

In this section $G = SL(3, k)$, and we assume $p > 3$. (For $p = 2$ the extensions between simple modules for $SL(3, 2)$ may be found in [5, Sect. 5]. For $p = 3$ one may also settle the question by ad hoc computations.)

4.1. Let $\lambda \in X_1(T)$ and write $\lambda = r\omega_1 + s\omega_2$, where ω_1 and ω_2 denote the two fundamental weights. If $a, b \in \mathbb{Z}$, then we write $a\omega_1 + b\omega_2 = (a, b)$. For instance, $\lambda = (r, s)$.

First we want to determine $\operatorname{Ext}^1_G (L(\nu), L(\lambda))$ for all p-bounded weights ν. It is well known that for this group $\dim \operatorname{Ext}^1_G (L(\nu), L(\lambda)) \leqslant 1$ for all such ν. In Table I we have given $\{\nu \mid \nu\ p\text{-bounded and } \operatorname{Ext}^1_G (L(\nu), L(\lambda)) \neq 0\}$ (this can be extracted, e.g., from [17] or [21]). Note that $\nu^1 \in \{0, \omega_1, \omega_2\}$ for all ν which appear. In particular, $\nu^1 \in C_0$ (even though we have not assumed $p > 3(h-1)$).

4.2. We now determine $\operatorname{Ext}^1_{SL(3,p)} (L(\mu), L(\lambda))$ for all $\mu, \lambda \in X_1(T)$.

Write $\lambda = (r, s)$ and $\mu = (t, u)$. We may assume that $t + u \leqslant r + s$, and then we prove that Theorem 3.2 holds by checking that Eq. 2 in Section 3.2 holds. Note that since $L(\nu^1)$ is at most three-dimensional, it is easy to determine $L(\nu^0) \otimes L(\nu^1)$ and $Q_1(\nu^0) \otimes L(\nu^1)$. Moreover, using (1) from Section 1, we get (compare [12, p. 53])

TABLE I

$\lambda = (r, s)$	ν
$r+s+2<p$	$(p-s-2, p-r-2), (p+r+s+1, p-s-2), (p-r-2, p+r+s+1)$
$r+s+2=p$	$(2p-1, r), (s, 2p-1)$
$r, s<p-1$ $r+s+2>p$	$(p-s-2, p-r-2), (2p-r-2, s+r+1-p), (s+r+1-p, 2p-s-2)$
$r=p-1$ $s<p-1$	$(s, 2p-s-2)$

$$Q_1(\lambda) = U_1(\lambda) \qquad \text{unless either } r=0 \text{ or } s=0$$
$$Q_1(0,s) = U_1(0,s) \oplus U_1(p-1,s) \qquad \text{for } s \neq 0$$
$$Q_1(r,0) = U_1(r,0) \oplus U_1(r,p-1) \qquad \text{for } r \neq 0$$
$$Q_1(0,0) = U_1(0,0) \oplus U_1(p-1,0) \oplus U(0,p-1) \oplus \mathrm{St}_1.$$

Let us illustrate the computations by an example. Consider the case where $r+s+2>p$ and $r,s<p-1$. Then one of the non-restricted weights v for which $L(v)$ extends $L(\lambda)$ (for G) is $v=(2p-r-2,\ s+r+1-p)$. We have (terms with a negative coordinate are understood to be 0)

$$L(v^0) \otimes L(v^1) = L(p-r-1, s+r+1-p) \oplus L(p-r-3, s+r+2-p) \oplus L(p-r-2, s+r-p)$$

and

$$Q_1(v^0) \otimes L(v^1) = Q_1(p-r-1, s+r+1-p) \oplus Q_1(p-r-3, s+r+2-p) \oplus Q_1(p-r-2, s+r-p),$$

where for the last statement we have used first the fact that $\dim \mathrm{Hom}_G(L(\omega), Q_1(v^0) \otimes L(v^1)) = \dim \mathrm{Hom}_G(L(\omega) \otimes L(v^{1*}), Q_1(v^0)) = [L(\omega) \otimes L(v^{1*}) : L(v^0)]_G$ and second the relations between Q_1 and U_1 stated above.

Hence we get contributions to $\mathrm{Ext}^1_{G(1)}(L(\mu), L(\lambda))$ for the following values of μ: $(p-r-1, s+r+1-p)$, $(p-r-3,\ s+r+2-p)$, and $(p-r-2,\ s+r-p)$.

The above procedure also works for $\lambda=(p-1,s)$, $s<p-1$, but $L(v^0) \otimes L(v^1)$ is not completely reducible in this case. More precisely, we find (for $v=(s, 2p-s-2)$)

$$L(v^0) \otimes L(v^1) = L(s+1, p-s-3) \oplus M,$$

where M is a G-module whose $G(1)$-socle is $L(s-1, p-s-2)$. However, we also get

$$Q_1(v^0) \otimes L(v^1) \cong Q_1(s+1, p-s-3) \oplus Q_1(s-1, p-s-2),$$

so that Eq. (2) in Section 3.2 still holds.

We have collected the results of our computations in Table II (recall that terms with a negative coordinate should be ignored).

In Table II we have omitted the case $s=p-1$, $r<p-1$, which is symmetric to the last row. Also, the case $r=s=p-1$ is not found because in this case $L(\lambda)=\mathrm{St}_1$ is injective, and hence $\mathrm{Ext}^1_{G(1)}(L(\mu), L(\lambda))=0$ for all μ.

The dimension of $\mathrm{Ext}^1_{G(1)}(L(\mu), L(\lambda))$ can also be read from the table,

TABLE II

$\lambda=(r,s)$	$\mu=(t,u)$ with $t+u\leqslant r+s$ and $\mathrm{Ext}^1_{G(1)}(L(\mu),L(\lambda))\neq 0$		
$r+s+2\leqslant p$		none	
$r+s+2>p$ $r,s<p-1$	$(p-s-2,p-r-2)$	$(p-r-1,s+r+1-p)$ $(p-r-3,s+r+2-p)$ $(p-r-2,s+r-p)$	$(s+r+1-p,p-s-1)$ $(s+r+2-p,p-s-3)$ $(s+r-p,p-s-2)$
$r=p-1$ $0<s<p-1$	$(s+1,p-s-3)$ $(s-1,p-s-2)$		
$s=0$ $r=p-1$	$(1,p-3)$	$(0,p-1)$	

namely as the number of times μ occurs in the row corresponding to λ. Note that this number is $\leqslant 3$ with equality occurring for $\lambda=\frac{2}{3}(p-1,p-1)$ and $\mu=\frac{1}{3}(p+2,p+2)$.

Remark. It is interesting to compare our results with those of [7, Sect. 8], [16, Sect. 5], and [17, Chap. 7], where the structure of the principal series modules for $SL(3,k)$ is discussed. Note in particular that not all extensions of $L(\lambda)$ occur in the $G(1)$-module induced from λ.

4.3. Let π denote the graph automorphism on A_2 which permutes the two simple roots. Then $G_\pi(1)=SU(3,p)$. Using the same techniques as in 4.2, we can compute the extensions between simple modules for $G_\pi(1)$ (see Section 3.3). The analogue of Eq. (2) in Section 3.2 is

$$\mathrm{Hom}_G(L(\mu),L(v^0)\otimes L(\pi(v^1)))=\mathrm{Hom}_G(L(\mu),Q_1(v^0)\otimes L(\pi(v^1)))$$

and the decomposition of Q_1's is given by

$$Q_1(r,s)=U_1(r,s)\qquad \text{unless } r=0 \text{ and } s<p-1 \text{ or vice versa}$$

$$Q_1(0,s)=U_1(0,s)\oplus U_1(p-1,s+1),\qquad 0\leqslant s<p-1$$

$$Q_1(r,0)=U_1(r,0)\oplus U_1(p-1,r+1),\qquad 0\leqslant r<p-1$$

$$Q_1(0,0)=U_1(0,0)\oplus U_1(p-1,1)\oplus U_1(1,p-1)\oplus \mathrm{St}_1.$$

This gives the results shown in Table III. In this table we have omitted the case symmetric to the last row, and we have not listed λ's for which $\mathrm{Ext}^1_{G_\pi(1)}(L(\mu),L(\lambda))=0$ for all μ with $t+u\leqslant r+s$.

Remark. Note that no simple module for $SL(3,p)$ or $SU(3,p)$ extends itself.

5. $Sp(4,p)$

In this section $G=Sp(4,k)$ and we assume $p>5$.

TABLE III

$\lambda=(r,s)$	$\mu=(t,u)$ with $t+u\leqslant r+s$ and $\mathrm{Ext}^1_{G_\pi(1)}(L(\mu),L(\lambda))\neq 0$		
$r=0, s=p-2$	$(p-2,0)$		
$r+s+2>p$		$(p-r-2, s+r+2-p)$	$(s+r+2-p, p-s-2)$
	$(p-s-2, p-r-2)$	$(p-r-1, s+r-p)$	$(s+r-p, p-s-1)$
$r, s<p-1$		$(p-r-3, s+r+1-p)$	$(s+r+1-p, p-s-3)$
$r=p-1$	$(s-1, p-s-1)$		
$s<p-1$	$(s, p-s-3)$		

5.1. Let α and β denote the two simple roots with α short. Then we write $\lambda=(r,s)$ if $\lambda=r\omega_1+s\omega_2$, where ω_1 and ω_2 are the fundamental weights corresponding to α and β, respectively.

Again our first task is to find the p-bounded weights ν for which $\mathrm{Ext}^1_G(L(\nu),L(\lambda))\neq 0$. For p-regular weights this can be done by computing the second layer in the Jantzen filtration for $H^0(\nu)$ because this coincides with the second socle layer (see [4]). (As all multiplicities of composition factors in the $H^0(\nu)$'s in question are 1, the so-called Jantzen conjecture holds for B_2, and we may use [4].) For p-singular weights we have used ad hoc computations.

Our results are stated in Table IV.

TABLE IV

$\lambda=(r,s)$	ν p-bounded with $\mathrm{Ext}^1_G(L(\nu),L(\lambda))=k$	
$r+2s+3<p$	$(r, p-r-s-3), (2p-r-1, p-s-2), (2p-r-2s-4, p+r+s+1)$	
$r+2s+3=p$	$(p+2s+2, p-s-2), (p-1, 2p-s-2)$	
$p<r+2s+3$, $r+s+2<p$	$(r, p-r-s-3), (2p-r-2s-4, s), (r+2s+2, p-s-2), (2p-r-2, s+r+1)$	
$r+s+2=p$	$(p+s, p-s-2), (2p-r-2, p-1)$	
$p<r+s+2$, $r+2s+3<2p$, $r<p-1$	$(2p-r-2s-4, s), (2p-r-2, s+r+1-p), (r, 2p-r-s-3), (r+2s+2, p-s-2)$	
$r=p-1$, $2s+2<p$	$(p-1, p-s-2)$	$(p+2s+1, p-s-2)$
$r+2s+3=2p$	$(2s+1, p-s-2)$	$(2p-1, p-s-2)$
$r+2+s>2p$, $r, s<p-1$	$(r, 2p-r-s-3), (2p-r-2, s+r+1-p), (r+2s+2-2p, 2p-s-2)$	
$r=p-1$, $2s+2>p$, $s<p-1$	$(2s+1-p, 2p-s-2)$	
$s=p-1$, $r<p-1$	$(2p-r-2, r)$	

5.2. We now give the results for $\operatorname{Ext}^1_{G(1)}(L(\mu), L(\lambda))$ (again in the form of a table with $\mu=(t,u)$, $\lambda=(r,s)$, and $t+2u\leqslant r+2s$). The method of computation is the same as the one used in Section 4, and we find that Theorem 3.2 holds for all λ, μ. It should be pointed out that there are cases where Eq. (2) in Section 3.2 fails. If for instance $r=0$, $p-3<2s<2p-4$, $t=2s+3-p$, $u=p-s-3$, and $v=(2s+2, p-s-2)$, then

$$\operatorname{Hom}_G(L(\mu), L(v^0)\otimes L(v^1))=0,$$

whereas

$$\operatorname{Hom}_G(L(\mu), Q_1(v^0)\otimes L(v^1))=\operatorname{Hom}_G(L(\mu)\otimes L(v^1), Q_1(v^0))=k^2.$$

However, if we interchange the roles of μ and λ, then Eq. (2) in Section 3.2 holds, and since $\operatorname{Ext}^1_{G(1)}(L(\mu), L(\lambda)\cong \operatorname{Ext}^1_{G(1)}(L(\lambda^*), L(\mu^*))=\operatorname{Ext}^1_{G(1)}(L(\lambda), L(\mu))$, this will settle the case. The same argument applies to all other cases where Eq. (2) in Section 3.2 fails.

Via Eq. (1) in Section 1 we obtain the following decomposition of the $Q_1(\lambda)$'s (compare [23, Table I].

$$
\begin{aligned}
Q_1(r,s)&=U_1(r,s) && \text{if } r,s>0\\
Q_1(0,s)&=U_1(0,s)\oplus U_1(p-1,s) && \text{if } s=(p-3)/2\\
Q_1(0,s)&=U_1(0,s)\oplus U_1(p-1,s)\\
&\quad\oplus U_1(p-1,s+1), && \text{if } 0<s<p-2,\ s\neq(p-3)/2\\
Q_1(0,s)&=U_1(0,s)\oplus U_1(p-1,s)\oplus 3\,\mathrm{St}_1 && \text{if } s=p-2\\
Q_1(0,p-1)&=U_1(0,p-1)\oplus \mathrm{St}_1\\
Q_1(r,0)&=U_1(r,0)\oplus U_1(r,p-1)\\
&\quad\oplus U_1(r+2,p-1), && \text{if } 0<r<p-3\\
Q_1(r,0)&=U_1(r,0)\oplus U_1(r,p-1)\oplus 3\,\mathrm{St}_1 && \text{if } r=p-3\\
Q_1(r,0)&=U_1(r,0)\oplus U_1(r,p-1), && p-2\leqslant r\leqslant p-1\\
Q_1(0,0)&=U_1(0,0)\oplus U_1(p-1,0)\\
&\quad\oplus U_1(p-1,1)\oplus U_1(0,p-1)\\
&\quad\oplus U(2,p-1)\oplus \mathrm{St}_1.
\end{aligned}
$$

In Table V, terms with a negative coordinate should as usual be ignored. It should be noted that there is some redundancy in the table because we did not follow the rule of listing only μ's with $t+2u\leqslant r+2s$ in the cases where this equality holds for some but not all (r,s) in the given range.

TABLE V

$\lambda=(r,s)$	$\mu=(t,u)$		
$r+2s+3\leqslant p$		none	
$r+2s+3>p$ $r+s+2<p$ $r>0$	$(r,p-r-s-3)$	$(r+2s+3-p,p-s-2)$ $(r+2s+1-p,p-s-1)$ $(r+2s+3-p,p-s-3)$ $(r+2s+1-p,p-s-2)$	
$2s+3>p$ $s+2<p$ $r=0$	$(0,p-s-3)$	$(2s+3-p,p-s-2)$ $(2s+1-p,p-s-1)$	
$r+s+2=p$		$(s+1,p-s-3)$ $(s-1,p-s-1)$ $(s-1,p-s-2)$	
$r+s+1=p$ $p-2>s>0$	$(2p-r-2s-4,s)$	$(p-r-1,0)$ $(p-r-3,1)$ $(p-r-3,0)$	$(r+2s+3-p,p-s-2)$ $(r+2s+1,p-s-1)$ $(r+2s+3-p,p-s-3)$
$r+s+1>p$ $r+2s+3<2p$ $r<p-1$	$(2p-r-2s-4,s)$	$(p-r-1,s+r+1-p)$ $(p-r-3,s+r+2-p)$ $(p-r-1,s+r-p)$ $(p-r-3,s+r+1-p)$	$(r+2s+3-p,p-s-2)$ $(r+2s+1-p,p-s-1)$ $(r+2s+3-p,p-s-3)$ $(r+2s+1-p,p-s-2)$
$r=p-1$ $2s<p-1$		$(2s+2,p-s-3)$ $(2s,p-s-2)$	
$r+2s+3=2p$		$(2s+2-p,p-s-3)$ $(2s-p,p-s-2)$	$(p-2,p-s-1)$ $(p-2,p-s-2)$
$r+2s+2=2p$ $r,s<p-1$	$(r,2p-r-s-3)$	$(p-r-1,s+r+1-p)$ $(p-r-s,s+r+2-p)$	$(0,p-s-1)$ $(2,p-s-3)$ $(0,p-s-2)$ $(0,p-s-3)$
$r+2s+2>2p$ $r,s<p-1$	$(r,2p-r-s-3)$	$(p-r-1,s+r+1-p)$ $(p-r-3,s+r+2-p)$ $(p-r-1,s+r-p)$ $(p-r-s,s+r+1-p)$	$(r+2s+2-2p,p-s-1)$ $(r+2s+4-2p,p-s-3)$ $(r+2s+2-2p,p-s-2)$ $(r+2s-2p,p-s-1)$ $(r+2s+2-2p,p-s-3)$
$r=p-1$ $p<2s+2<2p$			$(2s+3-p,p-s-3)$ $(2s+1-p,p-s-2)$ $(2s+3-p,p-s-1)$ $(2s+1-p,p-s-3)$
$s=p-1$ $r<p-1$		$(p-r-1,r-1)$ $(p-r-s,r+1)$ $(p-r-3,r)$	

Note. "Selfextensions" exist for $\lambda\in\{(r,(p-3)/2)\mid r>0\}\cup\{(r,(p-1)/2)\mid r<p-1\}$ (compare [13]).

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